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SEMICONTINUOUS PROCESSES IN MULTI-DIMENSIONAL EXTREME VALUE THEORY

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The structure of the large values attained by a stationary random process indexed by a one-dimensional parameter is well described in the literature in many cases of interest. Here this structure is described in terms of semicontinuous processes. The main advantage with this is that it automatically generalizes to processes with multi-dimensional parameter. Concrete asymptotic results are given for Gaussian fields, which, in case of continuous parameter, may possess very erratic sample paths.

extreme values * random fields * semicontinuous processes * Gaussian fields

1. Introduction

This paper provides a new framework for weak convergence of extremes and, in doing so, it extends some well-known results in the one-dimensional theory to random fields. The emphasis is on the continuous-parameter case, although new results are obtained also for processes with discrete parameter. The main tools are some recent developments of the theories of semicontinuous processes and random sets. Cf. Vervaat [23] and Norberg [19, 20]. For background information on random sets the reader is referred to Matheron [18]. All the one-dimensional results for extremes which the present work extends can be found in a monograph by Leadbetter, Lindgren and Rootzén [15]. Accordingly this is our main reference and its comprehensive list of references is recommended to the reader interested in original papers.

We proceed to discuss the main result of this paper. Let $Y=\{Y(s), s\in R_+^d\}$ be a real-valued stationary random process indexed by a d-dimensional parameter ($R_+=[0,\infty)$, $d\in N=\{1,2,\ldots\}$). Suppose the trajectories of Y are continuous. Whenever $T>0$ let $c_T:R\to R$ be increasing and right continuous ($R=(-\infty,\infty)$). Let further $-\infty\leqslant c\leqslant \inf_{T,x} c_T(x)$. Write $\mathscr{B}$ for the collection of all bounded (i.e. relatively compact) Borel sets in R_+^d. For $T>0$ define

$$X_T(s)=c_T(Y(sT)),\quad s\in R_+^d, \tag{1.1}$$

$$\xi_T(B)=\sup_{s\in B} c_T(Y(sT)),\quad B\in\mathscr{B}, \tag{1.2}$$

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(convention: $\sup_{s\in\emptyset} X_T(s)=c$) and

$$\varphi_T=\{(s,x)\in R_+^d\times(c,\infty],\, x\leq c_T(Y(sT)\}. \tag{1.3}$$

The X_T's are semicontinuous processes [23] on R_+^d, the ξ_T's are (maxitive) random capacities [20] on R_+^d and the φ_T's are random sets [18, 19] in $R_+^d\times(c,\infty]$ ($(c,\infty]$ is the usual right-end compactification of (c,∞) and $R_+^d\times(c,\infty]$ is endowed with the product topology). We refer to φ_T as the hypograph of X_T.

Note that, for $T>0$, $K\subseteq R_+^d$ compact and $c<x\leq\infty$,

$$\{X_T(s)<x \text{ for all } s\in K\}=\{\xi_T(K)<x\}=\{\varphi_T\cap K\times[x,\infty]=\emptyset\}. \tag{1.4}$$

Cf. [20]. All these events are measurable. This is a straightforward consequence of the fact that all trajectories of Y are continuous.

We study the asymptotic distributions of the processes X_T, ξ_T and φ_T as $T\to\infty$ in a case which has been extensively studied when $d=1$. It is known [20] that a limit result for one of them automatically converts into limit results for the others.

We now introduce our class of limit processes. Let ρ be a Poisson process on $R_+^d\times(c,\infty]$ with intensity $E\rho=\lambda\times\mu$, where λ is d-dimensional Lebesgue measure while μ is a non-zero measure on $(c,\infty]$ satisfying

$$\mu[x,\infty]<\infty, \quad x>c. \tag{1.5}$$

Introduce a random capacity

$$\xi(B)=\sup\{x,\, x>c,\, \rho(B\times[x,\infty])\geq 1\}, \quad B\in\mathscr{B}. \tag{1.6}$$

Then, for $B\in\mathscr{B}$ and $x>c$,

$$\{\xi(B)<x\}=\{\rho(B\times[x,\infty])=0\}. \tag{1.7}$$

The probability of the latter event is $F(x)^{\lambda B}$, where

$$F(x)=\exp(-\mu[x,\infty]), \quad x>c, \tag{1.8}$$

is a (left-continuous) distribution function on $(c,\infty]$. Moreover ξ has independent peaks in the sense that $\xi(B_1),\ldots,\xi(B_n)$ are independent whenever $n\in N$ and $B_1,\ldots,B_n\in\mathscr{B}$ are (pairwise) disjoint. Put further

$$X(s)=\xi(\{s\}), \quad s\in R_+^d, \tag{1.9}$$

and write φ for the hypograph of X, i.e.

$$\varphi=\{(s,x)\in R_+^d\times(c,\infty],\, x\leq X(s)\}. \tag{1.10}$$

The semicontinuous process X, defined in (1.9), is a rather peculiar process. Note that $X(s)=c$ as for all fixed $s\in R_+^d$, while it is far from true that the event on which $X(s)=c$ for all $s\in R_+^d$ has probability one. In fact the latter can occur if, and only if, μ is identically zero—a case that we have excluded.

Let us note that, for $K\subseteq R_+^d$ compact and $x>c$,

$$\{X(s)<x \text{ for all } s\in K\}=\{\xi(K)<x\}=\{\varphi\cap K\times[x,\infty]=\emptyset\}. \tag{1.11}$$

We have already seen a similar statement for X_T, and it is perhaps not surprising that the following three assertions, suitably interpreted, are equivalent:

$$X_T \xrightarrow{\mathrm{d}} X, \tag{1.12}$$

$$\xi_T \xrightarrow{\mathrm{d}} \xi, \tag{1.13}$$

$$\varphi_T \xrightarrow{\mathrm{d}} \varphi. \tag{1.14}$$

We write $\xrightarrow{\mathrm{d}}$ for convergence in distribution, i.e. weak convergence of the corresponding probability measures. Cf. Billingsley [4]. The topological spaces involved in assertions (1.12)-(1.14) are defined in Section 2.

In the main result of this paper we present conditions on Y and the c_T's under which the equivalent assertions (1.12)-(1.14) hold for an appropriate choice of the vertical intensity μ.

We now present some tractable necessary and sufficient conditions for (1.12)-(1.14). First conclude from [20] that (1.13) holds iff

$$(\xi_T(B_1), \dots, \xi_T(B_n)) \xrightarrow{\mathrm{d}} (\xi(B_1), \dots, \xi(B_n)) \tag{1.15}$$

whenever $n \in N$ and $B_i \in \mathscr{B}$, $\lambda \partial B_i = 0$, $1 \leq i \leq n$. (We write B^- for the closure, B^0 for the interior and $\partial B = B^- \backslash B^0$ for the boundary of B.) The space underlying the weak convergence in (1.15) is $[c, \infty]^n$ equipped with the product topology.

Next we remark that (1.14) holds iff

$$\lim_T P\{\varphi_T \cap B \neq \emptyset\} = P\{\varphi \cap B \neq \emptyset\} \tag{1.16}$$

for all bounded $B \subseteq R_+^d \times (c, \infty]$ with

$$P\{\varphi \cap B^- \neq \emptyset,\ \varphi \cap B^0 = \emptyset\} = 0 \tag{1.17}$$

[19] ($B \subseteq R_+^d \times (c, \infty]$ is bounded iff $B \subseteq K \times [x, \infty]$ for some compact $K \subseteq R_+^d$ and some $x > c$).

In general there is no direct characterization known of (1.12) in terms of X and the X_T's. However it is shown in [20] that if $c = 0$, and X and the X_T's are finite valued, then (1.12) holds iff

$$\sup_s f(s) X_T(s) \xrightarrow{\mathrm{d}} \sup_s f(s) X(s) \tag{1.18}$$

for all continuous and compactly supported $f: R_+^d \to R_+$. We write $\mathscr{C}_+$ for the collection of all such functions.

We now discuss some cases of particular interest. First let

$$c_T = 1_{[u_T, \infty)}, \quad T > 0. \tag{1.19}$$

(1_A denotes the indicator function of A.) Here u_T is some high level typically increasing with T, and the objects of interest are the normalized excursion sets

$$\eta_T = \{X_T \geq 1\} = \{s \in R_+^d,\ Y(sT) \geqslant u_T\}, \quad T > 0. \tag{1.20}$$

Note that the η_T's are random sets in R_+^d.

Put further

$$\eta = \{X \geqslant 1\}. \tag{1.21}$$

Note that the random set η is the support of a stationary Poisson process on R_+^d with intensity $\tau = \mu[1, \infty]$. It is shown in [20] that (1.12)-(1.14) hold iff

$$\eta_T \xrightarrow{\mathrm{d}} \eta. \tag{1.22}$$

We may conclude from [19] that (1.22) is equivalent to

$$\lim_T P\{\eta_T \cap B \neq \emptyset\} = P\{\eta \cap B \neq \emptyset\}, \quad B \in \mathscr{B},\ \lambda \partial B = 0. \tag{1.23}$$

Of course

$$P\{n \cap B = \emptyset\} = \exp(-\tau\lambda B), \quad B \in \mathscr{B}, \tag{1.24}$$

while, for $K \subseteq R_+^d$ compact,

$$P\{\eta_T \cap K = \emptyset\} = P\{\sup_{s/T \in K} Y(s) < u_T\}, \quad T > 0. \tag{1.25}$$

Moreover it is proved in [20] that (1.22) is equivalent to

$$\sup_{s \in \eta_T} f(s) \xrightarrow{\mathrm{d}} \sup_{s \in \eta} f(s), \quad f \in \mathscr{C}_+. \tag{1.26}$$

Clearly $s \in \eta_T$ iff $Y(sT) \geq u_T$. Let us also note here that (1.22) implies

$$\partial \eta_T \xrightarrow{\mathrm{d}} \eta. \tag{1.27}$$

This is proved in Proposition 2.2.

Now consider the case $d = 1$. Suppose the event that η_T contains no isolated points and $\partial \eta_T$ is locally finite has probability one for each fixed $T > 0$. This can be shown to hold under regularity conditions on Y similar to those in the first two sections of Chapter 7 in [15], and it implies that the points of $\partial \eta_T$ are either up- or down-crossings for $\{Y(sT), s \geqslant 0\}$ of the level u_T. Write $\partial^+ \eta_T$ for the subset of up-crossing points. Then

$$\partial^+ \eta_T \xrightarrow{\mathrm{d}} \eta, \tag{1.28}$$

is a rather straightforward consequence of (1.27). Of course a corresponding limiting result holds for down-crossings.

In this paper $\#A$ denotes cardinality of a set A. Moreover, when ξ is a locally finite random set, then $\#\xi$ denotes the point process defined by $\#\xi(K) = \#(\xi \cap K)$. Consequently, $\#\partial^+\eta_T$ as a random measure is the point process of up-crossings of the level u_T made by the normalized process $\{Y(sT), s \geq 0\}$. Assume

$$\limsup_T E\#\partial^+\eta_T[s, t) \leq \tau(t-s), \quad 0 \leq s < t < \infty. \tag{1.29}$$

Then $\{\#\partial^+\eta_T\}$ is a relatively compact collection of point processes on R_+, and we may conclude from [11] that (1.28) implies

$$\#\partial^+\eta_T \xrightarrow{\mathrm{d}} \#\eta. \tag{1.30}$$

General conditions on Y and the u_T's for (1.30) to hold can be found in [15]. Therefore this theme will not be pursued further here.

Next consider the case

$$c_T = \sum_{i=1}^{n} 1_{[u_{iT}, \infty)}, \quad T > 0, \tag{1.31}$$

where $n \in N$ and $u_{1T} \leq \cdots \leq u_{nT}$. For every level $i \in \{1, \ldots, n\}$ we introduce the excursion sets

$$\eta_{iT} = \{X_T \geq i\} = \{s \in R_+^d,\ Y(sT) \geq u_{iT}\}, \quad T > 0, \tag{1.32}$$

and write

$$\eta_i = \{X \geq i\}. \tag{1.33}$$

It follows from [20] that (1.12)-(1.14) hold iff

$$(\eta_{1T}, \ldots, \eta_{nT}) \xrightarrow{\mathrm{d}} (\eta_1, \ldots, \eta_n). \tag{1.34}$$

Of course there are characterizations of (1.34) which are similar to the characterizations (1.23) and (1.26) of (1.22). Furthermore, the arguments leading to (1.27) and, for $d = 1$, to (1.28) and (1.30) are easily extended to joint consideration of several levels as above.

Let us finally consider the case

$$c_T(x) = a_T(x - b_T), \quad x \in R,\ T > 0. \tag{1.35}$$

Here $a_T > 0$ and $b_T \in R$, $T > 0$. Assume (1.12)-(1.14). Then, by (1.15),

$$a_T(\sup_{0 \leq s \leq T} Y(s) - b_T) \xrightarrow{\mathrm{d}} F. \tag{1.36}$$

(Of course, for $s = (s_1, \ldots, s_d) \in R^d$, $0 \leq s \leq T$ iff $0 \leq s_k \leq T$, $1 \leq k \leq d$.) Routine argumentation involving the asymptotic independence in (1.15) now show that F must belong to one of the three classes of max-stable distributions (see [15]), provided F is non-degenerate of course. So the general results also imply the classical theorem of extremal types. Its formulation is left to the reader.

In particular such an F must be continuous. Hence, by Proposition 2.2, for $n \in N$ and $-\infty < x_1 \leqslant \cdots \leqslant x_n < \infty$,

$$(\{X_T \geqslant x_i\}, 1 \leqslant i \leqslant n) \xrightarrow{\mathrm{d}} (\{X \geqslant x_i\}, 1 \leqslant i \leqslant n). \tag{1.37}$$

This is (1.34) with $u_{iT} = x_i/a_T + b_T$, $1 \leqslant i \leqslant n$, $T > 0$. Hence all previous results on excursion sets and their boundaries continue to hold with this choice of $\{c_T, T > 0\}$ (provided (1.12)-(1.14) hold, of course).

Although the main emphasis is on continuous-parameter processes, we also present a couple of basic results for discrete-parameter processes. They lead to a complete description of the extremes for a large class of vector-valued processes on Z_+^d ($Z_+ = \{0, 1, \ldots\}$), which we now are going to sketch.

Fix d, $m \in N$ and let $X_n = \{(X_{nj}^1, \ldots, X_{nj}^m), j \in Z_+^d\}$ be a stationary random field for each $n \in N$. Then, under suitable restrictions on the X_n's, the point process on $R_+^d \times (-\infty, \infty]$ supported by the sets

$$\{(j/n, X_{nj}^i), j \in Z_+^d\}, \quad 1 \leqslant i \leqslant m, \tag{1.38}$$

are, in the limit as $n \to \infty$, independent Poisson processes with intensities $\lambda \times \mu_i$, $1 \leqslant i \leqslant m$, where λ is d-dimensional Lebesgue measure as above and the μ_i's satisfy (1.5).

We now describe the organization of the paper. Section 2 presents some background material on random sets and semicontinuous processes, to make the paper reasonably self-contained. There we also prove a few new results which we believe to be of independent interest. Section 3 discusses the extremal theory for processes with discrete parameter. This section further serves as a preparation for the technically more complicated continuous-parameter theory in Section 4. Finally, in Section 5 we apply the results to Gaussian processes.

Most of the notation is introduced where needed. A few conventions follow here. A collection $\mathscr{A}$ of subsets of a locally compact topological space S is called *separating* if, for each instance of compact K and open G in S with $K \subseteq G$ there is an $A \in \mathscr{A}$ such that $K \subseteq A \subseteq G$. The set R^d is endowed with the coordinatewise partial order. For $s = (s_1, \ldots, s_d) \in R_+^d$ we write $[0, s]$ for the rectangle $\prod_i [0, s_i]$ and put $\|s\| = \max\{s_1, \ldots, s_d\}$. Whenever a scalar occurs in a formula at the location of a vector, it means the corresponding vector with equal components. So $s + x$ with $s \in R^d$ and $x \in R$ denotes the vector $(s_1 + x, \ldots, s_d + x)$.

2. Random capacities, semicontinuous processes and random sets

Here the theory needed to understand assertions such as (1.12)-(1.14) and their consequences is reviewed and developed further.

Let S be a locally compact second countable Hausdorff space and fix $c \in [-\infty, \infty)$. Write $\mathscr{K}$, $\mathscr{G}$ and $\mathscr{F}$, resp, for the classes of compact, open and closed subsets of S. Furthermore write $\mathscr{B}$ for the class of all bounded Borel sets in S. The letters K, G,

F and B, with or without subscripts, are in the following reserved for members of $\mathscr{K}$, $\mathscr{G}$, $\mathscr{F}$ and $\mathscr{B}$, resp. Moreover, the letters x and s denote generic elements of $(c, \infty]$ and S, resp., unless stated otherwise.

Let us say that $f: \mathscr{K} \cup \mathscr{G} \to [c, \infty]$ is a *capacity* and write $f \in \mathscr{U}_c$ if

$$f(\emptyset) = c, \tag{2.1}$$

$$f(A_1) \leq f(A_2), \quad A_1, A_2 \in \mathscr{K} \cup \mathscr{G}, \ A_1 \subseteq A_2, \tag{2.2}$$

$$f(G) = \lim_n f(G_n), \quad G, G_1, G_2, \ldots \in \mathscr{G}, \ G_n \uparrow G, \tag{2.3}$$

$$f(K) = \lim_n f(K_n), \quad K, K_1, K_2, \ldots \in \mathscr{K}, \ K_n \downarrow K. \tag{2.4}$$

Note that, for $f \in \mathscr{U}_c$,

$$f(G) = \sup_{K \subseteq G} f(K), \quad G \in \mathscr{G}, \tag{2.5}$$

$$f(K) = \inf_{K \subseteq G} f(G), \quad K \in \mathscr{K}. \tag{2.6}$$

The verification of these facts is straightforward. We further say that a capacity $f \in \mathscr{U}_c$ is *regular* if

$$\sup_{K \subseteq B} f(K) = \inf_{B \subseteq G} f(G), \quad B \in \mathscr{B}. \tag{2.7}$$

All regular capacities extend to $\mathscr{B}$ by means of the formula

$$f(B) = \sup_{K \subseteq B} f(K), \quad B \in \mathscr{B}. \tag{2.8}$$

Endow $\mathscr{U}_c$ with the topology generated by the families

$$\{f \in \mathscr{U}_c, f(K) < x\}, \quad K \in \mathscr{K}, \ x > c,$$

and

$$\{f \in \mathscr{U}_c, f(G) > x\}, \quad G \in \mathscr{G}, \ x > c.$$

It will be referred to as the *vague* topology. It is proved in [20] that $\mathscr{U}_c$ is compact, second countable and Hausdorff. Hence $\mathscr{U}_c$ can be given a complete and separable metric compatible with the topology.

Recall that $f: S \to [c, \infty]$ is called *upper semicontinuous* (usc) if $\{f \geq x\} \in \mathscr{F}$ whenever $c < x \leq \infty$. Write $\mathscr{F}_c$ for the collection of all such function. Sometimes we write $\mathscr{F}_+$ for $\mathscr{F}_0$.

Whenever $f \in \mathscr{F}_c$ we introduce a regular capacity $f^\vee$ and a closed subset of $S \times (c, \infty]$, denoted hypo(f) and called the *hypograph* of f, by means of the formulae

$$f^\vee(B) = \sup_{s \in B} f(s), \quad B \in \mathscr{B}, \tag{2.9}$$

$$\text{hypo}(f) = \{(s, x), x \leq f(s)\} \tag{2.10}$$

(convention in (2.9) and in similar definitions below: $\sup_{s \in \emptyset} f(s) = c$).

Note that, for $f_1,\dots,f_n \in \mathscr{F}_c$,

$$\text{hypo}(\sup_i f_i) = \bigcup_i \text{hypo}(f_i), \tag{2.11}$$

while, for $\{f_\alpha\} \subseteq \mathscr{F}_c$,

$$\text{hypo}(\inf_\alpha f_\alpha) = \bigcap_\alpha \text{hypo}(f_\alpha)\,. \tag{2.12}$$

Thus, hypo preserves the lattice structure of $\mathscr{F}_c$.

Note that $F \to 1_F$ is an embedding of $\mathscr{F}$ into $\mathscr{F}_+$. Clearly

$$1_F^\vee(B) = \begin{cases} 1, & \text{if } F \cap B \neq \emptyset, \\ 0, & \text{otherwise.} \end{cases} \tag{2.13}$$

We may now conclude that the induced vague topology on $\mathscr{F}$ is generated by the families

$$\{F, F \cap K = \emptyset\}, \quad K \in \mathscr{K},$$

and

$$\{F, F \cap G \neq \emptyset\}, \quad G \in \mathscr{G}.$$

Thus it concides with the topology discussed in [18]. Endow $\mathscr{F}$ with this topology. (This sentence and similar ones should be interpreted as follows: Whenever we meet a collection of closed subsets of a locally compact second countable Hausdorff space, we assume it is endowed with this topology.)

It is not hard to see that the vague topology on $\mathscr{U}_c$ coincides with the coarsest topology making the mappings $f \to f(K)$ and $f \to f(G)$ upper and lower semicontinuous resp. However note that, for $f \in \mathscr{F}_c$,

$$f^\vee(K) < x \quad \text{iff} \quad \text{hypo}(f) \cap K \times [x, \infty] = \emptyset$$

while

$$f^\vee(G) > x \quad \text{iff} \quad \text{hypo}(f) \cap G \times (x, \infty] \neq \emptyset.$$

It is now rather a straightforward task to prove that the topologies on $\mathscr{F}_c$ induced by the mappings $\cdot^\vee$ and hypo coincide [20, 23]. In the literature this toplogy is sometimes referred to as the *hypo* topology. Let us also note here that the relative hypo topology on the collection of all usc functions on S into $[0, \infty)$ is generated by the mappings

$$f \to \sup_s f(s) g(s), \quad g \in \mathscr{C}_+ \tag{2.14}$$

[20]. Here and below $\mathscr{C}_+$ denotes the collection of all compactly supported continuous functions on S into R_+.

Random elements in $\mathscr{U}_c$, $\mathscr{F}_c$ and $\mathscr{F}$ are subsequently called *random capacities*, *semicontinuous processes* and *random sets*, resp. Cf. [18, 19, 20, 23].

Let ξ be a semicontinuous process on S. Then $\text{hypo}(\xi)$ is a random set in $S\times(c,\infty]$. By [18, p. 9] so is also $\partial\,\text{hypo}(\xi)$. The *excursion set* $\{\xi\geq x\}$ is a random set in S. This follows from the fact that $\{\xi\geq x\}\cap K=\emptyset$ iff $\xi(K)<x$. Note that $\{\xi\geq x+n^{-1}\}$ converges pointwise to $\{\xi>x\}^-$ as $n\to\infty$. Hence $\{\xi>x\}^-$ is a random set in S and the event on which $\{\xi\geq x\}=\{\xi>x\}^-$ is measurable. By [18, p. 47] we conclude that the event that all excursion sets are locally finite is measurable. On this event we have $\partial\{\xi\geq x\}=\{\xi\geq x\}$ for all $x>c$ and, moreover, $\partial\,\text{hypo}(\xi)=\text{hypo}(\xi)$. If it has probability one then $\{\xi>x\}$ is a random set, since $\{\xi>x\}=\{\xi>x\}^-$ a.s. and $\{\xi>x\}\cap B\neq\emptyset$ iff $\xi^\vee(B)>x$. Let us finally note here that the evaluation $\xi^\vee(B)$ is a random variable in $[c,a]$ for all $B\in\mathscr{B}$, if the sample space on which ξ is defined is complete [20]. In this case it follows that $\{\xi\geq x\}\cap B\neq\emptyset$ is a measurable event. Clearly this fact holds for arbitrary random sets too, cf. [18].

We proceed to discuss convergence in distribution of random elements of $\mathscr{U}_c$, $\mathscr{F}_c$ and $\mathscr{F}$. First, suppose $\xi,\xi_1,\xi_2,\ldots$ are random capacities on S. Then, by [20], $\xi_n\xrightarrow{d}\xi$ iff

$$(\xi_n(B_1),\ldots,\xi_n(B_m))\xrightarrow{d}(\xi(B_1),\ldots,\xi(B_m)) \tag{2.15}$$

whenever $m\in N$ and $B_i\in\mathscr{B}$, $\xi(B_i^0)=\xi(B_i^-)$ a.s., $1\leq i\leq m$. In (2.15) $\xrightarrow{d}$ is w.r.t. the product topology on $[c,\infty]^m$, $\xi_n(B_i)$ denotes an arbitrary random variable satisfying

$$\xi_n(B_i^0)\leq\xi_n(B_i)\leq\xi_n(B_i^-)\quad\text{a.s.}$$

and

$$\xi(B_i)=\xi(B_i^0)=\xi(B_i^-)\quad\text{a.s.}$$

Next suppose $\xi,\xi_1,\xi_2,\ldots$ are random sets in S. By [19], $\xi_n\xrightarrow{d}\xi$ iff

$$\lim_n P\{\xi_n\cap B\neq\emptyset\}=P\{\xi\cap B\neq\emptyset\} \tag{2.16}$$

for all $B\in\mathscr{B}$ with $P\{\xi\cap B^0=\emptyset,\ \xi\cap B^-\neq\emptyset\}=0$. Moreover, by [20], $\xi_n\xrightarrow{d}\xi$ iff

$$\sup_{s\in\xi_n}f(s)\xrightarrow{d}\sup_{s\in\xi}f(s),\quad f\in\mathscr{C}_+. \tag{2.17}$$

Finally let ξ and the ξ_n's be semicontinuous processes on S. Obviously $\xi_n\xrightarrow{d}\xi$ iff $\xi_n^\vee\xrightarrow{d}\xi^\vee$ iff $\text{hypo}(\xi_n)\xrightarrow{d}\text{hypo}(\xi)$. Moreover, if ξ and the ξ_n's take their values in $[0,\infty)$ then $\xi_n\xrightarrow{d}\xi$ iff

$$\sup_s f(s)\xi_n(s)\xrightarrow{d}\sup_s f(s)\xi(s),\quad f\in\mathscr{C}_+. \tag{2.18}$$

This fact is proved in [20]. Note also that, for S discrete, $\xi_n\xrightarrow{d}\xi$ iff

$$(\xi_n(s_1),\ldots,\xi_n(s_m))\xrightarrow{d}(\xi(s_1),\ldots,\xi(s_m)),\quad m\in N,\ s_1,\ldots,s_m\in S. \tag{2.19}$$

Now we proceed with some new consequences of the assertion $\xi_n\xrightarrow{d}\xi$. They are needed in the following sections. However we do believe that they have some independent value too.

Proposition 2.1. *Let $\xi, \xi_1, \xi_2, \ldots$ be semicontinuous processes on S into $[c, \infty)$ and assume $\partial\,\mathrm{hypo}(\xi_n) \xrightarrow{d} \partial\,\mathrm{hypo}(\xi)$. Then $\xi_n \xrightarrow{d} \xi$. Conversely, suppose $\xi_n \xrightarrow{d} \xi$, that $\partial\,\mathrm{hypo}(\xi) = \mathrm{hypo}(\xi)$ a.s., and that S is locally connected. Then $\partial\,\mathrm{hypo}(\xi_n) \xrightarrow{d} \mathrm{hypo}(\xi)$.*

Proposition 2.2. *Let $\xi, \xi_1, \xi_2, \ldots$ be semicontinuous processes on S and suppose $\xi_n \xrightarrow{d} \xi$. Fix $m \in N$ and $x_i > c$ with $\{\xi \geqslant x_i\} = \{\xi > x_i\}^-$ a.s., $1 \leqslant i \leqslant m$. Then*

$$(\{\xi_n \geqslant x_1\}, \ldots, \{\xi_n \geqslant x_m\}) \xrightarrow{d} (\{\xi \geqslant x_i\}, \ldots, \{\xi \geqslant x_m\}). \tag{2.20}$$

Moreover the set of points $x > c$ for which $\{\xi \geqslant x\} = \{\xi > x\}^-$ a.s. is dense. Suppose further that S is locally connected and that

$$\partial\{\xi \geqslant x_i\} = \{\xi \geqslant x_i\} \text{ a.s.}, \quad 1 \leqslant i \leqslant m.$$

Then

$$(\partial\{\xi_n \geqslant x_i\}, 1 \leqslant i \leqslant m) \xrightarrow{d} (\{\xi \geqslant x_i\}, 1 \leqslant i \leqslant m). \tag{2.21}$$

Proof of Proposition 2.1. First note that, for $f \in \mathscr{F}_c$,

$$f^\vee(K) < x \quad \text{iff} \quad \mathrm{hypo}(f) \cap K \times [x, \infty] = \emptyset$$

and

$$f^\vee(B) > x \quad \text{iff} \quad \mathrm{hypo}(f) \cap B \times (x, \infty] \neq \emptyset$$

[20]. Now let $f, f_1, f_2, \ldots \in \mathscr{F}_c$ take their values in $[c, \infty)$. Suppose $\partial\,\mathrm{hypo}(f_n) \to \partial\,\mathrm{hypo}(f)$. If $f^\vee(K) < x$ then $\partial\,\mathrm{hypo}(f) \cap K \times [x, \infty] = \emptyset$. Hence, for n sufficiently large, $\partial\,\mathrm{hypo}(f_n) \cap K \times [x, \infty] = \emptyset$. Of course the latter implies $f_n^\vee(K) < x$. Next, if $f^\vee(G) > x$ then $\partial\,\mathrm{hypo}(f) \cap G \times (x, \infty] \neq \emptyset$. Hence $\partial\,\mathrm{hypo}(f_n) \cap G \times (x, \infty] \neq \emptyset$, and therefore $f_n^\vee(G) > x$, for sufficiently large n. We may now conclude that $f_n \to f$.

Conversely, suppose $f_n \to f$, that $\partial\,\mathrm{hypo}(f) = \mathrm{hypo}(f)$ and that S is locally connected. Then also $S \times (c, \infty]$ is locally connected and we may conclude by [18, p. 9] that the mapping ∂ is lsc. In the terminology of [18] we now get

$$F = \partial F \subseteq \liminf \partial F_n \leqslant \limsup \partial F_n \leqslant \lim F_n = F. \tag{2.22}$$

Here we have written $F = \mathrm{hypo}(f)$ and $F_n = \mathrm{hypo}(f_n)$, $n \in N$. Hence $\partial\,\mathrm{hypo}(f_n) \xrightarrow{d} \mathrm{hypo}(f)$.

By [4, Theorem 5.1] the proposition now follows. $\square$

Proof of Proposition 2.2. Let $\mathscr{G}_0$ be a countable open base for $\mathscr{G}$. Suppose $\{\xi \geqslant x\} \neq \{\xi > x\}^-$. Then $\xi(s) \geqslant x$ for some $s \notin \{\xi > x\}^-$. But then $\xi^\vee(G) \leqslant x$ for some $G \in \mathscr{G}_0$ with $s \in G$. Hence $\xi^\vee(G) = x$. We may now conclude that

$$\{x > c, P\{\{\xi \geqslant x\} = \{\xi > x\}^-\} < 1\} \subseteq \bigcup_{G \in \mathscr{G}_0} \{x > c, P\{\xi(G) = x\} > 0\}. \tag{2.23}$$

On the right hand side we have a countable union of countable sets. Hence the set on the left hand side in (2.23) is countable. This proves the middle assertion.

Now let $f, f_1, f_2, \ldots \in \mathcal{F}_c$. Suppose $f_n \to f$ and that $\{f \geq x\} = \{f > x\}^-$. A simple argument yields at once $\{f_n \geq x\} \to \{f \geq x\}$. Suppose $\partial\{f \geq x\} = \{f \geq x\}$ too. The mapping ∂ is continuous at F if $F = \partial F$ [18, p. 9]. Hence $\partial\{f_n \geq x\} \to \{f \geq x\}$.

By [4, Theorem 5.1] the remaining assertions of the proposition now follow. □

Let ξ be a semicontinuous process on S having, with probability one, locally finite excursion sets. Then $\#\{\xi \geq c\}$ is a point process on S for each $x > c$. Note that $x \to \#\{\xi \geq x\}$ is decreasing and left continuous. It is not hard to see that there is a unique point process $|\xi|$ on $S \times (c, \infty]$ satisfying

$$|\xi| K \times [x, \infty] = \#\{\xi \geq x\}(K), \quad K \in \mathcal{K}, \; x > c, \tag{2.24}$$

(i.e. $|\xi| K \times [x, \infty]$ equals the cardinality of the set $\{\xi \geq x\} \cap K$). We may say that $|\xi|$ counts all peaks of ξ, while $\#\{\xi \geq x\}$ only counts the peaks above the level x.

Proposition 2.3. *Let $\xi, \xi_1, \xi_2, \ldots$ be semicontinuous processes on S with locally finite excursion sets, and let $\mathcal{A} \in \mathcal{B}$ be a semi-ring. Suppose the ring generated by $\mathcal{A}$ is separating. Let further $D \subseteq (c, \infty)$ be dense. If $|\xi_n| \xrightarrow{d} |\xi|$ then $\xi_n \xrightarrow{d} \xi$. Conversely, $|\xi_n| \xrightarrow{d} |\xi|$ if $\xi_n \xrightarrow{d} \xi$ and*

$$\limsup_n E\#\{\xi_n \geq x\}(A) \leq E\#\{\xi \geq x\}(A), \quad A \in \mathcal{A}, \; x \in D. \tag{2.25}$$

Proof. First note that the first assertion is a trivial consequence of [4, Theorem 5.1]. Assume $\xi_n \xrightarrow{d} \xi$ and (2.25). By dominated convergence the latter extends to all $x > c$. By Proposition 2.2,

$$(\{\xi_n \geq x_1\}, \ldots, \{\xi_n \geq x_m\}) \xrightarrow{d} (\{\xi \geq x_1\}, \ldots, \{\xi \geq x_m\}) \tag{2.26}$$

whenever $m \in N$ and $\{\xi \geq x_i\} = \{\xi > x_i\}^-$ a.s., $1 \leq i \leq m$. It is now a rather obvious consequence of [11, Theorem 4.7, Exc. 4.14] that

$$(\#\{\xi_n \geq x_1\}, \ldots, \#\{\xi_n \geq x_m\}) \xrightarrow{d} (\#\{\xi \geq x_1\}, \ldots, \#\{\xi \geq x_m\}) \tag{2.27}$$

whenever m and the x_i's are as above. By Proposition 2.2 the set of points $x > c$ for which $\{\xi \geq x\} = \{\xi > x\}^-$ is dense in (c, ∞). Hence it contains a countable dense subset Q. Introduce

$$\mathcal{B}_Q = \bigcap_{x \in Q} \{B, \#\{\xi \geq x\}(\partial B) = 0 \text{ a.s.}\}. \tag{2.28}$$

It can be shown that $\mathcal{B}_Q$ is a separating ring (cf. [11, lemma 4.2], which shows that every set in this intersection is a separating ring). Hence the class of finite unions of sets $B \times [x, y)$ or $B \times [x, \infty]$, where $B \in \mathcal{B}_Q$ and $x, y \in Q$, is a separating ring. Denote it by $\mathcal{R}$. From (2.27) we now get, for $m \in N$ and $R_1, \ldots, R_m \in \mathcal{R}$,

$$(|\xi_n| R_1, \ldots, |\xi_n| R_m) \xrightarrow{d} (|\xi| R_1, \ldots, |\xi| R_m). \tag{2.29}$$

Hence $|\xi_n| \xrightarrow{d} |\xi|$. Cf. [20]. □

Let ξ be a semicontinuous process on S into $[0,\infty]$ with locally finite excursion sets. Then

$$|\xi|_p K = \int_{(0,\infty)} x \,\mathrm{d}|\xi|(K, x) = \int x 1_K(s) \,\mathrm{d}|\xi|(s, x) \tag{2.30}$$

is an extended valued random variable for all $K \in \mathscr{K}$. If the event that these variables are finite has probability one, then, clearly, (2.30) defines a random measure on S. For $f \in \mathscr{S}_+$—the collection of all Borel measurable non-negative functions on S—we get by a routine approximation procedure

$$|\xi|_p f = \int f \,\mathrm{d}|\xi|_p = \int x f(s) \,\mathrm{d}|\xi|(s, x). \tag{2.31}$$

A semicontinuous process ξ on S into $[c,\infty]$ is said to have *independent peaks* [20, 22] if $\xi^\vee(B_1), \ldots, \xi^\vee(B_n)$ are independent whenever $B_1, \ldots, B_n \in \mathscr{B}$ are disjoint. Let ξ be such a process. Then, by [20], either $\xi(s) = \infty$ for all $s \in S$ a.s. or there exists some $h \in \mathscr{F}_c$ which is finite in at least one point and some locally finite measure m on $S \times (c, \infty] \setminus \text{hypo}(h)$, such that

$$P \bigcap_{i=1}^{n} \{\xi^\vee(K_i) < x_i\} = \exp\left(-m \bigcup_{i=1}^{n} K_i \times [x_i, \infty]\right) \tag{2.32}$$

whenever $n \in N$ and $K_i \in \mathscr{K}$, $x_i > h^\vee(K_i)$, $1 \leqslant i \leqslant n$, and, for $K \in \mathscr{K}$ and $c < x \leqslant h^\vee(K)$,

$$P\{\xi^\vee(K) < x\} = 0. \tag{2.33}$$

Suppose $h(s) = c$, $s \in S$. Then ξ has locally finite excursion sets. Moreover, $|\xi|$ and $\#\{\xi \geqslant x\}$ are Poisson processes on $S \times (c, \infty]$ and S, resp, with intensities m and $B \to mB \times [x, \infty]$, $B \in \mathscr{B}$.

If $c = 0$ and m is concentrated on $S \times (0, \infty)$ such that

$$\int_{(0,\infty)} x \,\mathrm{d}m(K, x) < \infty, \quad K \in \mathscr{K}, \tag{2.34}$$

then $|\xi|_p K < \infty$ for all $K \in \mathscr{K}$ with probability one, so $|\xi|_p$ is a well-defined random measure on S. It is easily seen that $|\xi|_p$ has independent increments and Laplace transform

$$E \exp(-|\xi|_p f) = \exp\left(-\int (1 - \mathrm{e}^{-xf(s)}) \,\mathrm{d}m(s, x)\right), \quad f \in \mathscr{S}_+. \tag{2.35}$$

Proposition 2.4. *Let $\xi, \xi_1, \xi_2, \ldots$ be semicontinuous processes on S into R_+. Suppose the sets $\{\xi_n > 0\}$ are locally finite with probability one, and that ξ satisfies* (2.32) *for some locally finite measure m concentrated on $S \times (0, \infty)$ such that* (2.34) *holds. Furthermore suppose $\xi_n \xrightarrow{\mathrm{d}} \xi$ and* (2.25), *so that $|\xi_n| \xrightarrow{\mathrm{d}} |\xi|$. If*

$$\lim_{k\to\infty} \limsup_n P\{|\xi_n| K \times (0, \infty) > k\} = 0, \quad K \in \mathscr{K}, \tag{2.36}$$

then $|\xi_n|_p \xrightarrow{\mathrm{d}} |\xi|_p$.

Proof. Fix $u \in U = \{x > 0, \{\xi \geq x\} = \{\xi > x\} \text{ a.s.}\}$. For $f \in \mathscr{F}_+$ we put $f_u = f 1_{[u,\infty)}(f)$. Note that $f_u \in \mathscr{F}_+$ and that $f_u^\vee(K) < x$ iff $f^\vee(K) < u$ or $u \leq f^\vee(K) < x$. Thus the mapping $f \to f_u$ is measurable. It is continuous at f if f has locally finite excursion sets and $\{f \geq u\} = \{f > u\}$. Now we conclude by [4, Theorem 5.1] that $\xi_{nu} \xrightarrow{\text{d}} \xi_u$, and, by Proposition 2.3,

$$|\xi_{nu}| \xrightarrow{\text{d}} |\xi_u|. \tag{2.37}$$

Now let $h : S \to R_+$ be continuous with compact support H and fix $v > u$. By (2.37), $X_{nv} \xrightarrow{\text{d}} X_v$, where

$$X_{nv} = \int_{S\times(0,\infty)} h(s)(x 1_{(0,v)}(x) + v 1_{[v,\infty)}(x)) \,\text{d}|\xi_{nu}|(s,x), \tag{2.38}$$

$$X_v = \int_{S\times(0,\infty)} h(s)(x 1_{(0,v)}(x) + v 1_{[v,\infty)}(x)) \,\text{d}|\xi_u|(s,x). \tag{2.39}$$

By monotone convergence,

$$X_v \xrightarrow{\text{d}} |\xi_u|_p h \tag{2.40}$$

as $v \to \infty$. Moreover, by (2.37),

$$\begin{aligned} \lim \operatorname{limsup} P\{|\xi_{nu}|_p h - X_{nv} > 0\} &\leq \lim_v \operatorname*{limsup}_n P\left\{\int_{S\times[v,\infty)} x h(s) \,\text{d}|\xi_{nu}|(s,x) > 0\right\} \\ &\leq \lim_v \operatorname*{limsup}_n P\{|\xi_{nu}| H \times [v,\infty] \geq 1\} \\ &\leq \lim_v P\{|\xi_u| H \times [v,\infty] \geq 1\} = 0. \end{aligned} \tag{2.41}$$

By [5, Theorem 25.5],

$$|\xi_{nu}|_p h \xrightarrow{\text{d}} |\xi_u|_p h. \tag{2.42}$$

Let $u \to 0$ through U. By monotone convergence,

$$|\xi_u|_p h \xrightarrow{\text{d}} |\xi|_p h. \tag{2.43}$$

Fix $\varepsilon > 0$. Then, by (2.36),

$$\begin{aligned} &\lim_u \operatorname*{limsup}_n P\{|\xi_n|_p h - |\xi_{nu}|_p h \geq \varepsilon\} \\ &\quad\leq \lim_u \operatorname*{limsup}_n P\{|\xi_n| H \times (0,u) \geq \varepsilon/(u \sup h)\} \\ &\quad= 0. \end{aligned} \tag{2.44}$$

By a second reference to [5, Theorem 25.5] we now see that

$$|\xi_n|_p h \xrightarrow{\text{d}} |\xi|_p h. \tag{2.45}$$

We conclude that $|\xi_n|_p \xrightarrow{\mathrm{d}} |\xi|_p$. $\square$

We conclude this section with a sufficient condition for convergence in distribution of a sequence $\{\xi_n\}$ of semicontinuous processes on S, in case the limiting process ξ has independent peaks and satisfies (2.32) for some locally finite measure m on $S\times(c,\infty]$. Let $\mathscr{A}\subseteq\mathscr{B}$ be a semi-ring whose generated ring is separating, and let $D\subseteq(c,\infty)$ be dense. Then $\xi_n \xrightarrow{\mathrm{d}} \xi$ if, for all $A\in\mathscr{A}$ and $x\in D$,

$$P\{\xi_n^\vee(A)\leq x\}\to\exp(-mA\times(x,\infty]), \tag{2.46}$$

and, whenever $k\in N$, $A_1,\ldots,A_k\in\mathscr{A}$ are disjoint and $x_1,\ldots,x_k\in D$,

$$P\bigcap_{j=1}^{k}\{\xi_n^\vee(A_j)\leq x_j\}-\prod_{j=1}^{k}P\{\xi_n^\vee(A_j)\leq x_j\}\to 0. \tag{2.47}$$

This follows easily from general sufficient conditions for convergence in distribution of semi-continuous processes given in [20].

In the sequel we will use the same notation for a semicontinuous process and its associated random capacity. So $\xi(K)=\sup_{s\in K}\xi(s)$ for semicontinuous processes ξ.

3. Discrete parameter random fields

This section discusses the theory of extremes for stationary random fields indexed by vectors of integers.

Fix d, $m\in N$ and let $X_n=\{(X_{nj}^1,\ldots,X_{nj}^m), j\in Z_+^d\}$ be a stationary random field for every $n\in N$. Let $c\in[-\infty,\infty)$ be fixed and, for $1\leq l\leq m$, assume $v_n^l\to\infty$ as $n\to\infty$. We introduce semicontinuous processes on R_+^d by means of

$$\xi_n^l(s)=\begin{cases}\max\{X_{nj}^l,c\} & \text{if } sv_n^l=j\in Z_+^d,\\ c & \text{otherwise.}\end{cases} \tag{3.1}$$

Further put

$$\xi_n(l,s)=\xi_n^l(s),\quad l\in\{1,\ldots,m\},\quad s\in R_+^d. \tag{3.2}$$

Clearly $\{\xi_n\}$ is a sequence of semicontinuous processes on $\{1,\ldots,m\}\times R_+^d$. The notation from Section 2 is retained with $S=R_+^d$. Thus, for example, $\mathscr{K}$ denotes the collection of all compact subsets of R_+^d.

Let λ be d-dimensional Lebesgue measure, and introduce independent semicontinuous processes $\xi^1,\ldots,\xi^m$ on R_+^d into $[c,\infty)$ with independent peaks satisfying

$$-\log P\bigcap_{i=1}^{k}\{\xi^l(K_i)\leq x_i\}=\lambda\times\mu_l\bigcup_{i=1}^{k}K_i\times(x_i,\infty) \tag{3.3}$$

whenever $k\in N$, $K_i\in\mathscr{K}$, $x_i>c$, $1\leq i\leq k$, $1\leq l\leq m$, where, for $1\leq l\leq m$, μ_l is a measure on (c,∞) with $\mu_l(x,\infty)<\infty$ for all $x>c$. Cf (2.32). Furthermore put

$$\xi(l,s)=\xi^l(s),\quad 1\leq l\leq m,\ s\in R_+^d. \tag{3.4}$$

Before discussing the main result of this section, let us note that

$$\xi_n \xrightarrow{\mathrm{d}} \xi \quad \text{iff} \quad (\xi_n^1, \ldots, \xi_n^m) \xrightarrow{\mathrm{d}} (\xi^1, \ldots, \xi^m),$$

that

$$|\xi_n| \xrightarrow{\mathrm{d}} |\xi| \quad \text{iff} \quad (|\xi_n^1|, \ldots, |\xi_n^m|) \xrightarrow{\mathrm{d}} (|\xi^1|, \ldots, |\xi^m|)$$

and that

$$|\xi_n|_p \xrightarrow{\mathrm{d}} |\xi|_p \qquad \text{iff} \quad (|\xi_n^1|_p, \ldots, |\xi_n^m|_p) \xrightarrow{\mathrm{d}} (|\xi^1|_p, \ldots, |\xi^m|_p).$$

The latter of course provided that $c = 0$ and that

$$\int_{(0,\infty)} x \, \mathrm{d}\mu_l(x) < \infty, \quad 1 \leq l \leq m. \tag{3.5}$$

This is an easy exercise which we leave to the reader.

Theorem 3.1. *With the notation introduced above, we have* $|\xi_n| \xrightarrow{\mathrm{d}} |\xi|$, *provided there is a dense* $D \subseteq (c, \infty)$ *such that*

$$\lim_n (v_n^l)^d P\{X_{n0}^l > x\} = \mu_l(x, \infty), \quad 1 \leq l \leq m, \ x \in D, \tag{3.6}$$

$$\lim_n (v_n^l)^d P\{X_{n0}^l > x, X_{n0}^k > x\} = 0, \quad 1 \leq k, l \leq m, \ k \neq l, \ x > c, \tag{3.7}$$

$$\lim_r \limsup_n r^d \sum_{i \neq j, i \leq v_n^k/r, j \leq v_n^l/r} P\{X_{ni}^k > x, X_{nj}^l > x\} = 0, \quad 1 \leq k, l \leq m, \ x > c, \tag{3.8}$$

and, for each $\beta > 0$, $s \in [0, \beta]^d$, $\gamma > 0$ *and* $D_0 \subseteq D$ *finite,*

$$\lim_n \sup \left| P \bigcap_{a=1}^{p} \{X_{ni_a}^{k_a} \leq x_a\} \cap \bigcap_{b=1}^{q} \{X_{nj_b}^{l_b} \leq y_b\} \right.$$
$$\left. - P \bigcap_{a=1}^{p} \{X_{ni_a}^{k_a} \leq x_a\} P \bigcap_{b=1}^{q} \{X_{nj_b}^{l_b} \leq y_b\} \right| = 0, \tag{3.9}$$

where the supremum extends over $p, q \in N$, $x_a, y_b \in D_0$, $1 \leq k_a, l_b \leq m$,

$$i_a / v_n^{k_a} \in [0, s - \gamma], \quad j_b / v_n^{l_b} \in [0, \beta]^d \setminus [0, s], \ 1 \leq a \leq p, \ 1 \leq b \leq q.$$

Suppose $c = 0$ *and* (3.5). *If, in addition to* (3.6)-(3.9),

$$\limsup_n (v_n^l)^d P\{X_{n0}^l > 0\} < \infty, \quad 1 \leq l \leq m, \tag{3.10}$$

then $|\xi_n|_p \xrightarrow{\mathrm{d}} |\xi|_p$.

In (3.8) we have written $\lim_r$ for $\lim_{r \to \infty}$. The proof of Theorem 3.1 is postponed. In applications one usually has $v_n^1 = \cdots = v_n^m$. We will need the extra generality in the next section dealing with the continuous parameter case.

Note that the mixing condition (3.9) is trivial and that (3.7)-(3.8) follow from (3.6) if the X^l_{nj}'s are independent for each fixed $n \in N$. In this case (3.6) is equivalent to

$$\lim_n P\left\{\sup_{j \leq v^l_n} X^l_{nj} \leq x\right\} = \exp(-\mu_l(x, \infty)), \quad 1 \leq l \leq m, \ x \in D, \tag{3.11}$$

provided

$$\lim_n P\{X^l_{n0} > x\} = 0, \quad 1 \leq l \leq m, \ x > c. \tag{3.12}$$

Condition (3.9) generalizes condition $D(u_n)$ in Leadbettter [12]. See also [13]. Other extensions of $D(u_n)$ which are special cases of (3.9) can be found in Adler [1], Davis [7, 8, 9], and Leadbetter, Lindgren and Rootzen [15]. These and many other authors discuss various aspects of the case where the sequence $\{X_n\}$ of stationary fields stems from a single stationary field $Y = \{(Y^1_j, \ldots, Y^m_j), j \in Z^d_+\}$ by means of

$$X^l_{nj} = c^l_n(Y^l_j), \quad j \in Z^d_+, \ 1 \leq l \leq m, \ n \in N, \tag{3.13}$$

where the c^l_n's are increasing functions on R. Condition (3.8) extends Leadbetter's $D'(u_n)$ [12]. Note that (3.7) is needed only if $m > 1$.

It can be seen from the proof below that, if (3.9) holds only for some fixed $\beta > 0$, then the assertions of the theorem are still valid provided we restrict ξ and the ξ_n's to $\{1, \ldots, m\} \times [0, \beta]^d$.

Proof of Theorem 3.1: Let $\mathscr{A}$ be the semi-ring of bounded rectangles in R^d_+. Note that, by (3.6),

$$E|\xi^l_{nx}|A = \sum_{j/v^l_n \in A} P\{X^l_{nj} \geq x\} \to \lambda A \mu_l[x, \infty) = E|\xi^l_x|A \tag{3.14}$$

for $1 \leq l \leq m$, $A \in \mathscr{A}$ and $x > c$ with $\mu_l\{x\} = 0$. Moreover, when $c = 0$,

$$\limsup_n P\{|\xi^l_n|A \times (0, \infty) > k\} \leq k^{-1} \limsup_n \sum_{j/v^l_n \in A} P\{X^l_{nj} > 0\}, \tag{3.15}$$

which tends to zero as $k \to \infty$ by (3.10). Thus, by Propositions 2.3 and 2.4, we only need to prove $\xi_n \xrightarrow{d} \xi$.

Lemma 3.2. *Under* (3.9) *and*

$$\limsup_n (v^l_n) P\{X^l_{n0} > x\} < \infty, \quad 1 \leq l \leq m, \ x > c, \tag{3.16}$$

the difference

$$P \bigcap_{j=1}^{k} \{\xi^l_n(A_j) \leq x_{lj}, l \in I_j\} - \prod_{j=1}^{k} P\{\xi^l_n(A_j) \leq x_{lj}, l \in I_j\} \tag{3.17}$$

tends to zero, whenever $k \in N$, $A_1, \ldots, A_k \in \mathscr{A}$, are disjoint, $I_1, \ldots, I_k \subseteq \{1, \ldots, m\}$ and $x_{lj} \in D$, $l \in I_j$, $1 \leq j \leq k$.

Proof. Clearly $\bigcup_j A_j \subseteq [0, \beta]^d$ for some $\beta > 0$. We may assume without loss of generality that

$$\{s \in R_+^d,\ s \leq t \text{ for some } t \in A_j\} \cap \bigcup_{i>j} A_i = \emptyset, \quad 1 \leq j < k. \tag{3.18}$$

For $1 \leq j \leq k$ write H_j for the event

$$\{\xi_n^l(A_j) \leq x_{lj},\ l \in I_j\}. \tag{3.19}$$

For fixed but arbitrary $\gamma > 0$, write further $A_j' = \{s \in A_j,\ s + \gamma \in A_j\}$, and H_j' for (3.19) with A_j' replacing A_j. A simple recursive argument yields

$$\left| P \bigcap_j H_j - \prod_j PH_j \right| \leq 2 \sum_{j<k} PH_j' \backslash H_j + \sum_{j<k} \left| PH_j' \cap \bigcap_{i>j} H_i - PH_j' P \bigcap_{i>j} H_i \right|. \tag{3.20}$$

The second sum on the right of (3.20) converges to zero by (3.9). The first sum on the right of (3.20) is bounded by

$$\sum_{j<k} \sum_{l \in I_j} P\{\xi_n^l(A_j \backslash A_j') > x_{lj}\} \leq \sum_{j<k} \sum_{l \in I_j} d(\gamma v_n^l + 1)(\beta v_n^l + 1)^{d-1} P\{X_{n0}^l > x_{lj}\}. \tag{3.21}$$

By taking limsup over n the expression on the right of (3.21) and then letting $\gamma \to 0$, we get zero by (3.16). This proves (3.17). $\square$

Proposition 3.3. *Suppose* (3.16) *and* (3.17), *and also*

$$P\{\xi_n^l([0, 1]^d) \leq x_l,\ l \in I\} \to \exp\left(-\sum_{l \in I} \mu_l(x_l, \infty) \right) \tag{3.22}$$

whenever $I \subseteq \{1, \ldots, m\}$ *and* $x_l \in D$ *for* $l \in I$. *Then* $\xi_n \xrightarrow{d} \xi$.

Proof. Write $\mathscr{A}_r$ for the rectangles in $\mathscr{A}$ with rational vertices. Fix $p_1, \ldots, p_d, q_1, \ldots, q_d \in N$, and put $p = \prod_i p_i$ and $q = \prod_i q_i$. By (3.17) and some simple estimation using stationarity and (3.16), we see that

$$P\{\xi_n^l([0, 1]^d) \leq x_l,\ l \in I\} - P\{\xi_n^l(\textstyle\prod_i [0, 1/p_i]) \leq x_l,\ l \in I\}^p, \tag{3.23}$$

$$P\{\xi_n^l(\textstyle\prod_i [0, q_i/p_i]) \leq x_l,\ l \in I\} - P\{\xi_n^l(\prod_i [0, 1/p_i]) \leq x_l,\ l \in I\}^q \tag{3.24}$$

both tend to zero. Thus, by (3.22),

$$P\left\{ \xi_n^l\left(\prod_i [0, q_i/p_i] \right) \leq x_l,\ l \in I \right\} \to \exp\left(-(q/p) \sum_{l \in I} \mu_l(x_l, \infty) \right). \tag{3.25}$$

By stationarity and (3.16), we now obtain

$$P\{\xi_n^l(A) \leq x_l,\ l \in I\} \to \exp\left(-\lambda A \sum_{l \in I} \mu_l(x_l, \infty) \right), \tag{3.26}$$

for all $A \in \mathscr{A}_r$, $I \subseteq \{1, \ldots, m\}$ and $x_l \in D$, $l \in I$.

The class of sets $\{l\}\times A$, where $1\leqslant l\leqslant m$ and $A\in\mathscr{A}_r$, is a semi-ring with a separating generated ring. By the discussion finishing Section 2, $\xi_n \xrightarrow{d} \xi$ now follows from (3.17) and (3.26). □

Clearly (3.16) is weaker than (3.6), so (3.22) remains to be proved. For this the full strength of (3.6)–(3.9) is needed.

Remainder of the proof of Theorem 3.1. Fix $r\in N$ and divide the unit cube $[0,1]^d$ into r^d equally large disjoint cubes. By stationarity and Lemma 3.2, we get

$$P\{\xi_n^l([0,1]^d)\leqslant x_l,\, l\in I\}-P\{\xi_n^l([0,1/r]^d)\leqslant x_l,\, l\in I\}^{r^d}\to 0. \tag{3.27}$$

By the inclusion-exclusion inequalities, we obtain

$$\begin{aligned}\sum_{l\in I}\sum_{0\leqslant j/v_n^l\leqslant 1/r} P\{X_{nj}^l>x_l\}-S_{I,r}&\leqslant 1-P\{\xi_n^l([0,1/r]^d)\leqslant x_l,\, l\in I\}\\ &\leqslant \sum_{l\in I}\sum_{0\leqslant j/v_n^l\leqslant 1/r} P\{X_{nj}^l>x_l\},\end{aligned} \tag{3.28}$$

where

$$\begin{aligned}S_{I,r}=&\sum_{k,l\in I,k\neq l,0\leqslant i/v_n^k,i/v_n^l\leqslant 1/r} P\{X_{ni}^k>x_k,\, X_{ni}^l>x_l\}\\ &+\sum_{k,l\in I,0\leqslant i/v_n^k,j/v_n^l\leqslant 1/r,i\neq j} P\{X_{ni}^k>x_k,\, X_{nj}^l>x_l\}.\end{aligned} \tag{3.29}$$

By stationarity and (3.6)–(3.8), we get

$$\begin{aligned}\sum_{l\in I}\mu_l(x_l,\infty)-o_r&\leqslant \liminf_n r^d(1-P\{\xi_n^l([0,1/r]^d)\leqslant x_l,\, l\in I\})\\ &\leqslant \limsup_n r^d(1-P\{\xi_n^l([0,1/r]^d)\leqslant x_l,\, l\in I\})\\ &\leqslant \sum_{l\in I}\mu_l(x_l,\infty),\end{aligned} \tag{3.30}$$

where $o_r\to 0$ as $r\to\infty$. By routine calculations (3.22) follows. This completes the proof of Theorem 3.1. □

Note that (3.17) and (3.22) follow from $\xi_n \xrightarrow{d} \xi$ whenever $\mu_l\{x\}=0$ for $1\leqslant l\leqslant m$ and $x\in D$.

We now discuss some applications of Theorem 3.1. Let $\{Y_j, j\in Z_+^d\}$ be a stationary process, let $D\subseteq R$ be dense and suppose there are constants $a_n>0$ and $b_n\in R$, such that

$$\lim_n P\{\sup_{0\leqslant j\leqslant n} Y_j\leqslant x/a_n+b_n\}=F(x),\quad x\in D, \tag{3.31}$$

for some non-degenerate distribution function F. Let $v_n^1=n$ and put for fixed $x\in D$,

$$X_{nj}^1=\begin{cases}1, & \text{if } Y_j>x/a_n+b_n\\ 0, & \text{otherwise.}\end{cases} \tag{3.32}$$

If (3.16) and (3.8) hold for arbitrary $x \in D$, then, by Proposition 3.3,

$$\lim_n P\{\sup_{0 \le j \le n} Y_j \le x/a_{nr} + b_{nr}\} = F(x)^{r^{-d}}, \quad x \in D, \ r \in N. \tag{3.33}$$

It follows that F is of extremal type, i.e. F must belong to one of the three classes of max-stable distributions. Cf. [15].

Let V_n be the number of independently and uniformly thrown arcs of length $1/n$, required to cover a given circle of unit circumference q times. Flatto [10] proves that

$$\lim_n P\{V_n \le n(x + \log n + q \log\log n)\} = \exp(-e^{-x}/(q-1)!), \quad x \in R. \tag{3.34}$$

Furthermore he gives a heuristic explanation of this fact, which, with the help of Theorem 3.1, may be made rigorous.

An application of the first assertion of Theorem 3.1 to the theory of thinning can be found in [6].

For $d = 1$ and $m \in N$, Berman has proved under conditions, distinct from (3.6)-(3.10), that

$$\left(\sum_{0 \le j \le n} X_{nj}^l, 1 \le l \le m\right) \xrightarrow{d} (Y_1, \ldots, Y_m), \tag{3.35}$$

where $Y_1, \ldots, Y_m$ are independent random variables with Laplace transforms

$$E \exp(-tY_l) = \exp\left(-\int_{(0,\infty)} (1 - e^{-tx})\, d\mu_l(x)\right), \quad t \ge 0, \ 1 \le l \le m, \tag{3.36}$$

[2, Theorem 5.1]. Berman's result may be extended to arbitrary $d \in N$. Note that (3.35) is a particular case of $|\xi_n|_p \xrightarrow{d} |\xi|_p$. Moreover, Berman's proof of (3.35) may be extended to a proof of $|\xi_n|_p \xrightarrow{d} |\xi|_p$. Thus this assertion holds under two different sets of conditions, between which the relation is not completely clear.

Let $\{Y_j, j \in N\}$ be a non-decreasing sequence of non-negative random variables satisfying

$$\lim_{j \to \infty} Y_j/j = \tau \quad \text{a.s.} \tag{3.37}$$

for some $\tau \in (0, \infty)$. Let $\eta_1, \eta_2, \ldots$ be random measures on $(0, \infty)$ and put

$$X_{nj} = X_{nj}^1 = \eta_n(Y_j, Y_{j+1}], \quad j \in N, n \in N. \tag{3.38}$$

Suppose the X_{nj}'s form a stationary sequence for each fixed $n \in N$. Suppose also (3.6)-(3.10). For Borel sets $B \subseteq (0, \infty)$, we put

$$\tilde{\eta}_n B = \eta_n\{sv_n, s \in B\}. \tag{3.39}$$

(Here and in similar situations below $v_n = v_n^1$.) By combining Theorem 1 of Lindvall [17] with our Theorem 3.1, we get $\tilde{\eta}_n \xrightarrow{d}$ some $\tilde{\eta}$ with Laplace transform

$$E \exp\left(-\int f\, d\tilde{\eta}\right) = \exp\left(-\tau^{-1} \int\int (1 - e^{-xf(s)})\, d\mu(x)\, d\lambda(s)\right),$$
$$f: (0, \infty) \to R_+ \quad \text{Borel measurable.} \tag{3.40}$$

This result extends Lindvall's Theorem 2, which treats the case when the X_{nj}'s are independent and identically distributed for each $n \in N$. Of course it is true also under the conditions of [2, Theorem 2.1].

The latter result may be used to analyze the extremal structure of processes $\rho = \{\rho_s, s > 0\}$, for which there exist random variables $0 \leq Y_1 < Y_2 < \cdots \to \infty$ a.s., making the sequence

$$\{(Y_{j+1} - Y_j), \rho_s,\ Y_j \leq s \leq Y_{j+1}; j \in N\} \tag{3.41}$$

stationary; cf. [21].

4. Continuous parameter

Fix d, $m \in N$ and let $\{X_n\}$ be a sequence of semicontinuous processes on $\{1, \ldots, m\} \times R^d_+$. Suppose the vector processes $(X^1_n, \ldots, X^m_n)$ are stationary when regarded as processes on the Borel sets of R^d_+. Continuous processes which are stationary in the usual sense are stationary also in this extended sense. Let the sequence $\{v_n\}$ be bounded away from zero, and define semicontinuous processes on $\{1, \ldots, m\} \times R^d_+$ by means of

$$\xi_n(l, s) = \xi^l_n(s) = X^l_n(sv_n) \vee c, \quad 1 \leq l \leq m,\ s \in R^d_+,\ n \in N. \tag{4.1}$$

In this section we shall discuss various aspects of convergence in distribution of ξ_n to the limit process ξ (see (3.4)).

Our first result gives sufficient conditions for $\xi_n \xrightarrow{d} \xi$. Here and below $\lim_\alpha$ denotes $\lim_{\alpha \to 0}$. For $1 \leq l \leq m$ put

$$F_l(x) = \exp(-\mu_l(x, \infty)), \quad x > c. \tag{4.2}$$

Theorem 4.1. *Let $D \subseteq (c, \infty)$ be dense, and suppose there exist numbers $0 < q^l_{\alpha n} = q^l_n \to 0$ as $n \to \infty$, for $1 \leq l \leq m$ and $\alpha > 0$, satisfying*

$$\lim \operatorname{limsup} P\{X^l_n([0, sv_n]) > x, X^l_n(jq^l_n) \leq x, 0 \leq jq^l_n \leq sv_n\} = 0,$$

$$s \in R^d_+,\ s > 0,\ 1 \leq l \leq m,\ x > c. \tag{4.3}$$

Suppose further that (3.9) *and* (3.16) *hold with X^l_{nj} and v^l_n respectively replaced by $X^l_n(jq^l_n)$ and v_n/q^l_n, for each $\alpha > 0$. If*

$$\begin{gathered}\lim_n P\{X^l_n([0, v_n]^d) \leq x_l, l \in I\} = \prod_{l \in I} F_l(x_l), \\ I \subseteq \{1, \ldots, m\}, \quad x_l \in D, l \in I,\end{gathered} \tag{4.4}$$

then $\xi_n \xrightarrow{d} \xi$.

Proof. Let $\mathscr{A}$ be as in the proof of Theorem 3.1. For $\alpha > 0$ and $n \in N$, let $\bar{\xi}_{\alpha n}(l, s) = \bar{\xi}^l_{\alpha n} = X^l_n(jq^l_n)$ if $sv_n/q^l_n = j \in Z^d_+$, $= c$ otherwise. Since $\bar{\xi}_{\alpha n} \leq \xi_n$ a.s. for $\alpha > 0$ and $n \in N$, (4.3) extends to

$$\lim_{\alpha} \limsup_{n} \left(P \bigcap_{j=1}^{k} \{\bar{\xi}^l_{\alpha n}(A_j) \leq x_{lj}, l \in I_j\} - P \bigcap_{j=1}^{k} \{\xi^l_n(A_j) \leq x_{lj}, l \in I_j\} \right) = 0, \tag{4.5}$$

where $k \in N$, $A_j \in \mathscr{A}$, $I_j \subseteq \{1, \ldots, m\}$, $x_{lj} > c$, $l \in I_j$, $1 \leq j \leq k$. Note that the assumption of stationarity in Lemma 3.2 may be weakened to

$$X^l_{nj} \stackrel{\mathrm{d}}{=} X^l_{n0}, \quad j \in Z^d_+, \ 1 \leq l \leq m, \ n \in N. \tag{4.6}$$

We see that (3.17) holds here too. Now $\xi_n \xrightarrow{\mathrm{d}} \xi$ follows as in the proof of Proposition 3.3. □

We proceed to discuss sufficient conditions for the sample path condition (4.3). As it applies for separate l, the dependence on l is suppressed in the notation.

Proposition 4.2. *Suppose there exist numbers $0 < q_{\alpha n} = q_n \to 0$ as $n \to \infty$ such that* (3.16) *holds with $X_n(jq_n)$ and v_n/q_n replacing X^l_{nj} and v^l_n respectively for every $\alpha > 0$. Suppose also that $v_n \to \infty$ as $n \to \infty$. If, for some $h > 0$,*

$$\limsup_{n} v_n^{d-1} P\{X_n([0, h]^d) > x\} = 0, \tag{4.7}$$

$$\lim_{\alpha} \limsup_{n} v_n^d P\{X_n([0, h]^d) > x, X_n(jq_n) \leq x, 0 \leq jq_n \leq h\} = 0 \tag{4.8}$$

hold whenever $x > c$, then (4.3) *holds. Moreover,* (4.7) *and* (4.8) *hold for arbitrary $h > 0$, if*

$$\limsup_{n} (v_n/q_n)^d P\{X_n([0, q_n]^d) > x\} < \infty, \tag{4.9}$$

$$\lim_{\alpha} \limsup_{n} (v_n/q_n)^d P\{X_n([0, q_n]^d) > x, X_n(jq_n) \leq x, 0 \leq j \leq 1\} = 0, \tag{4.10}$$

for all $x > c$.

Proof. Fix $s \in R^d_+$, $s > 0$, and $x > c$. By dividing the rectangle $[0, [sv_n/h + 1]h]$ into cubes with side h, we obtain the estimate

$$\begin{aligned} &P\{X_n([0, sv_n]) > x, X_n(jq_n) \leq x, 0 \leq jq_n \leq sv_n\} \\ &\quad \leq d(1 + h/q_n)^{d-1}(\|s\| v_n/h)^d P\{X_n(0) > x\} \\ &\qquad + (\|s\| v_n/h)^d P\{X_n([0, h]^d) > x, X_n(jq_n) \leq x, 0 \leq jq_n \leq h\} \\ &\qquad + d(1 + \|s\| v_n/h)^{d-1} P\{X_n([0, h]^d) > x\}, \end{aligned} \tag{4.11}$$

from which the first assertion follows. The second assertion follows similarly by dividing the cube $[0, h]^d$ into cubes with sides of length q_n. □

It is not hard to see that (4.4) follows from

$$\lim_n P\{X^l_n(jq^l_n) \le x_l, 0 \le jq^l_n \le v_n, l \in I\} = \prod_{l \in I} G_{\alpha l}(x_l),$$
$$I \subseteq \{1, \ldots, m\}, \quad x_l \in D, \quad l \in I, \quad \alpha > 0, \tag{4.12}$$

where the distribution functions $G_{\alpha l}$ are such that

$$\lim_\alpha G_{\alpha l}(x) = F_l(x), \quad x \in D, \ 1 \le l \le m. \tag{4.13}$$

The next result gives sufficient conditions for (4.12) in the case $m = 1$. The dependence on α and l is suppressed in the notation.

Proposition 4.3. *Let $D \subseteq (c, \infty)$ be dense. Suppose there are numbers $0 < q_n \to 0$ as $n \to \infty$, satisfying (3.9) and (3.16) with X^l_{nj} and v^l_n replaced by $X_n(jq_n)$ and v_n/q_n, resp. Suppose also that $v_n \to \infty$, and that*

$$\lim_r \limsup_n (v_n/q_n)^d \sum_{h < \|jq_n\| \le v_n/r} P\{X_n(0) > x, X_n(jq_n) > x\} = 0, \quad x > c, \tag{4.14}$$

for some $h > 0$ satisfying

$$\lim_n v_n^d P \bigcup_{0 \le jq_n \le kh} \{X_n(jq_n) > x\} = h^d \prod_i k_i(-\log G(x)),$$
$$x \in D, \quad k = (k_1, \ldots, k_d), \ k_i = 1, 2, \ 1 \le i \le d. \tag{4.15}$$

Then

$$\lim_n P\{X_n(jq_n) \le x, 0 \le jq_n \le v_n\} = G(x), \quad x \in D. \tag{4.16}$$

Proof. Fix $r \in N$. Conclude as in the proof of Theorem 3.1 that

$$P\{X_n(jq_n) \le x, 0 \le jq_n \le v_n\} - P\{X_n(jq_n) \le x, 0 \le jq_n \le v_n/r\}^{r^d} \to 0. \tag{4.17}$$

For $l \in Z^d_+$ put

$$H(l) = \bigcup_{l \le jq_n/h \le l+1} \{X_n(jq_n) > x\}. \tag{4.18}$$

By (3.16) with X_{n_j} and v_n respectively replaced by $X_n(jq_n)$ and v_n/q_n, we get

$$|P \bigcup_{0 \le 1h \le v_n/r} H(l) - P \bigcup_{0 \le jq_n \le v_n/r} \{X_n(jq_n) > x\}| \to 0. \tag{4.19}$$

The inclusion-exclusion inequalities show

$$\sum_{0 \le lh \le v_n/r} PH(l) - \sum_{0 \le kh, lh \le v_n/r, k \ne l} PH(k) \cap H(l)$$
$$\le P \bigcup_{0 \le lh \le v_n/r} H(l) \le \sum_{0 \le lh \le v_n/r} PH(l). \tag{4.20}$$

By (4.15) the extreme terms tend to $-(h/r)^d \log G(x)$ as $n\to\infty$. If $\|k-l\|>1$ then $PH(k)\cap H(l)$ is bounded by

$$(1+h/q_n)^d \sum_{h<\|jq_n\|\leqslant v_n/r} P\{X_n(0)>x, X_n(jq_n)>x\}. \tag{4.21}$$

Now suppose $\|k-l\|=1$. If the intersection of the cubes $[k, k+1]$ and $[l, l+1]$ has dimension $d-1$ then

$$\lim_n v_n^d PH(k)\cup H(l) = -2h^d \log G(x) \tag{4.22}$$

by (4.15). It follows

$$\lim_n v_n^d PH(k)\cap H(l) = 0. \tag{4.23}$$

If this intersection has lower dimension, then there are events $B(i)$ with $H(i)\subseteq B(i)$, $i=k, l$, satisfying

$$\lim_n v_n^d PB(i) = -2^{d-1}h^d \log G(x), \quad i=k, l, \tag{4.24}$$

$$\lim_n v_n^d PB(k)\cup B(l) = -2^d h^d \log G(x), \tag{4.25}$$

as can be seen from a simple geometric argument. So (4.23) holds in this case too. The number of terms with $\|k-l\|=1$ in the second sum in (4.20) is finite and independent of n. Hence

$$\lim_r \limsup_n r^d \sum_{0\leqslant kh, lh\leqslant v_n/r, k\neq l} PH(k)\cap H(l) = 0. \tag{4.26}$$

The proposition now follows by routine arguments. □

Theorem 4.1 together with Propositions 4.2 and 4.3 provides a method for proving that $\xi_n \xrightarrow{\text{d}} \xi$. This is illustrated in the next section.

The discussion in the introduction about the interpretation of $\xi_n \xrightarrow{\text{d}} \xi$ (see 1.13)) still applies, although the latter is proved here in a more general context.

Let us conclude this section by an estimate which shows that (1.28) follows from (1.27). Let s, $t\in R_+$, $s\leqslant t$ and fix $T>0$. Then

$$\begin{aligned} 0 &\leqslant P\{\partial\eta_T\cap[s, t]\neq\emptyset\} - P\{\partial^+\eta_T\cap[s, t]\neq\emptyset\} \\ &\leqslant P\{\partial\eta_T\cap[s, t]\neq\emptyset, \partial^+\eta_T\cap[s, t]=\emptyset\} \\ &\leqslant P\{s\in\eta_T\} = P\{Y(0)\geqslant u_T\}. \end{aligned} \tag{4.27}$$

5. Applications to Gaussian fields

Below, Φ, ϕ will denote the standard normal distribution function and density, resp. Recall that, as $u\to\infty$,

$$(1-\Phi(u))^{-1}\phi(u)/u \to 1. \tag{5.1}$$

The following result is useful also for non-Gaussian applications. Its proof is omitted.

Lemma 5.1. *Let u be an increasing left-continuous function on a closed set $V \subseteq R$. Suppose, in the case* $\inf V = -\infty$, *that*

$$\inf\{u(x), x \in V\} = -\infty, \tag{5.2}$$

and, in the case $\sup V = \infty$, *that*

$$\sup\{u(x), x \in V\} = \infty. \tag{5.3}$$

Define

$$c(y) = \sup\{x \in V, u(x) \leqslant y\}, \quad y \in R, \tag{5.4}$$

where $\sup \emptyset = \inf V$. *Then c is V-valued, increasing and right-continuous. Furthermore, for $y \in R$ and $x \in V$ with $x > \inf V$,*

$$c(y) \geqslant x \quad \text{iff} \quad y \geqslant u(x). \tag{5.5}$$

Note that the only requirement on u at the point $\inf V$ (provided $\inf V > -\infty$) is

$$u(\inf V) \leqslant \inf\{u(x), x \in V\}, \tag{5.6}$$

since u must be increasing. We may even have $u(\inf V) = -\infty$. In this case, (5.5) is trivially true for $x = \inf V$. For example, if u is an increasing function on $\{1, \ldots, n\}$ then we may extend u to $V = \{0, \ldots, n\}$ by putting $u(0) = -\infty$. In this case the lemma yields $c = \sum_{j>0} 1_{[u(j),\infty)}$.

Now consider a stationary Gaussian field $\{Y_j, j \in Z^d\}$ with zero mean, unit variance and covariances $r(j) = EY_iY_{i+j}$ satisfying

$$\sup_{j \neq 0} |r(j)| < 1. \tag{5.7}$$

Fix $\tau > 0$ and choose levels $u_n = u_n(\tau)$ such that

$$n^d \phi(u_n)/u_n \to \tau, \tag{5.8}$$

and assume

$$n^d \sum_{0<\|j\|\leqslant n} |r(j)| \exp(-u_n^2/(1+|r(j)|)) \to 0. \tag{5.9}$$

Let X_{nj} be 1 if $Y_j \geqslant u_n$, 0 otherwise. Put $c = 0$ and $v_n = n$. Condition (3.6) follows at once from (5.8), while (3.8) and (3.9) follow by a straightforward use of the Normal Comparison Lemma (i.e. Theorem 4.2.1 in [15]). The case $d = 1$ is treated in Lemma 4.4.1 of [15]. We conclude by Theorem 3.1 that the distribution of the normalized point process of exceedances of the level u_n, i.e. the counting measure of the set $\{j/n, j \in Z_+^d, Y_j \geqslant u_n\}$, converges weakly to a Poisson process on R_+^d with intensity τ.

Note that (5.8) holds iff

$$u_n^2 = -2\log\tau - \log 4\pi + 2\log n^d - \log\log n^d + o(1) \tag{5.10}$$

(cf. proof of [15, Theorem 1.5.3]). Thus we may always assume that u_n is a continuous function of τ.

Note also that (5.9) follows from (5.8) if (5.7) is replaced by either of

$$r(j)\log\|j\| \to 0 \quad \text{as } \|j\| \to \infty, \tag{5.11}$$

$$\sum_j |r(j)|^p < \infty \quad \text{for some } p > 0. \tag{5.12}$$

Cf. [15, Ch. 4.5], which treats the case $d=1$.

Next consider an R^m-valued stationary Gaussian field $\{(Y_j^1,\ldots,Y_j^m), j \in Z^d\}$ with means zero, unit variances and covariance $r_{kl}(j) = EY_i^k Y_{i+j}^l$ satisfying (5.11) or (5.12). Also suppose that $|r_{kl}(0)| < 1$ when $k \neq l$.

Fix $c \geq -\infty$. For $1 \leq l \leq m$ let $\tau_l : (c, \infty) \to R_+$ be decreasing, left-continuous and such that $\tau_l(x) \to 0$ as $x \to \infty$. Write μ_l for the measure on (c, ∞) with $\mu_l[x, \infty) = \tau_l(x)$, $x > c$. Let U_l be the support of μ_l. Note that U_l is closed in (c, ∞). Put $V_l = U_l \cup \{c\}$ if $c > -\infty$, $= U_l$ otherwise. Clearly V_l is closed in R. For $1 \leq l \leq m$ choose functions u_n^l on U_l such that (5.8) holds pointwise. If c is finite, extend u_n^l to V_l by putting $u_n^l(c) = -\infty$. Suppose u_n^l satisfies the assumptions of Lemma 5.1 and define c_n^l by (5.4). Put $X_{nj}^l = c_n^l(Y_j^l)$, $j \in Z_+^d$, and let $v_n^l = n$. Condition (3.6) follows as above at once from (5.8). Condition (3.7) follows by straightforward calculations, using the Normal Comparison Lemma, from the fact that $|r_{kl}(0)| < 1$ whenever $k \neq l$. Conditions (3.8) and (3.9) follow by calculations similar to the case $m=1$. So, by Theorem 3.1, the point processes supported by

$$\{(j/n, c_n^l(Y_j^l)), j \in Z_+^d, c_n^l(Y_j^l) > c\}, \quad 1 \leq l \leq m, \tag{5.13}$$

are asymptotically independent, with Poisson processes on $R_+^d \times (c, \infty)$ with intensities $\lambda \times \mu_l$, $1 \leq l \leq m$, as limits in distribution.

Note that no condition is needed on the cross-covariance in the case $m=2$ and $Y_j^2 = -Y_j^1$. Cf. Davis [7].

Also note that, in the classical case where $\tau_l(x) = e^{-x}$, $x \in R$, we may choose $u_n^l(x) = x/a_n + b_n$, where

$$a_n = (2\log n^d)^{1/2}, \tag{5.14}$$

$$b_n = a_n - (2a_n)^{-1}(\log\log n^d + \log 4\pi) \tag{5.15}$$

(cf. [15, Theorem 1.5.3]).

Now we turn our attention to the continuous-parameter case. Let $\{Y_s, s \in R^d\}$ be a stationary Gaussian field with zero mean, unit variance and covariances $r(s) = EY_tY_{t+s}$ satisfying

$$\sup_{\|s\| \geq h} |r(s)| < 1, \quad h > 0. \tag{5.16}$$

Also suppose there is a continuous non-zero function C on $R^d \setminus \{0\}$ such that, for some a with $0 < a \leq 2$,

$$(1 - r(sq))/q^a \to C(s) \quad \text{as } 0 < q \to 0, \tag{5.17}$$

for all $s \in R^d$, $s \neq 0$.

For each $\alpha > 0$, let $0 < q_\alpha(u) = q_\alpha$ be such that

$$u^{2/a} q_\alpha(u) \to \alpha \text{ as } u \to \infty, \tag{5.18}$$

and put

$$\psi(u) = H_a \phi(u) u^{2d/a-1}, \tag{5.19}$$

where

$$H_a = \lim_T T^{-d} \int_{(0,\infty)} e^x P\left\{ \sup_{0 \le s \le T} Z_s > x \right\} dx, \tag{5.20}$$

with $\{Z_s\}$ a Gaussian process having

$$EZ_s = C(s), \quad s \in R_+^d, \tag{5.21}$$

$$\operatorname{Cov}[Z_s, Z_t] = C(s) + C(t) - C(s-t), \quad s, t \in R_+^d. \tag{5.22}$$

Fix $h > 0$ and let $k = (k_i) \in Z^d$, $0 < k \le 2$. Then, as $u \to \infty$,

$$P\left\{ \max_{0 \le jq_\alpha \le kh} Y_{jq_\alpha} > u \right\} \Big/ \psi(u) \to h^d \left(\prod_i k_i \right) H_a(\alpha), \tag{5.23}$$

where $H_a(\alpha) \to 1$ as $\alpha \to 0$, and

$$P\left\{ \sup_{0 \le s \le h} Y_s > u \right\} \Big/ \psi(u) \to h^d. \tag{5.24}$$

This may be seen by a slight extension of the arguments yielding Lemmas 1 to 5 in Bickel and Rosenblatt [3]. See [15, Ch. 12] for the case $d = 1$.

Now fix $\tau > 0$ and choose $u_T = u_T(\tau)$ so that

$$\begin{aligned} &u_T^2 - 2 \log T^d - (2d - a) a^{-1} \log \log T^d \\ &\quad \to -2 \log \tau - \log 2\pi + 2 \log H_a + (2d - a) a^{-1} \log 2 \end{aligned} \tag{5.25}$$

as $T \to \infty$. Then

$$T^d \psi(u_T) \to \tau, \tag{5.26}$$

and, moreover,

$$u_T^2 / (2 \log T^d) \to 1. \tag{5.27}$$

So that if

$$q_T = q_{\alpha T} = \alpha (2 \log T^d)^{-1/a}, \tag{5.28}$$

then

$$u_T^{2/a} q_{\alpha T} \to \alpha. \tag{5.29}$$

By (5.23) and (5.24) we get, for $h > 0$,

$$T^d P\left\{ \max_{0 \le jq_T \le kh} Y_{jq_T} > u_T \right\} \to \tau h^d \left(\prod_i k_i \right) H_a(\alpha), \tag{5.30}$$

where $k=(k_i)\in Z^d$, $0<k_i\leq 2$, and

$$T^d P\left\{\sup_{0\leq s\leq h} Y_s > u_T\right\} \to \tau h^d, \tag{5.31}$$

resp. We conclude

$$T^d P\left\{\sup_{0\leq s\leq h} Y_s > u_T,\ Y_{jq_T}\leq u_T,\ 0\leq jq_T\leq h\right\} \to (1-H_a(\alpha)). \tag{5.32}$$

Note that $1-H_a(\alpha)\to 0$ as $\alpha\to 0$.

Let $X_T(s)=1$ if $Y_{sT}\geq u_T$, $=0$ otherwise. Suppose

$$\left(\frac{T}{q_T}\right)^d \sum_{h\leq\|jq_T\|\leq T} |r(jq_T)| \exp\left(-\frac{u_T^2}{1+|r(jq_T)|}\right)\to 0 \tag{5.33}$$

for $\alpha>0$ and $h>0$. Let $T\to\infty$ through the subsequence $\{T(n)\}$. Clearly (3.16) holds, with v_n^1 and X_{n0}^l replaced by $T(n)/q_{\alpha T(n)}$ and $X_{T(n)}(0)$, resp, for each $\alpha>0$. Condition (4.7) follows from (5.31), while (5.32) implies (4.8). By Proposition 4.2, the sample path condition (4.3) is at hand. That (3.9) holds, with v_n^l and X_{nj}^l replaced by $T(n)/q_{\alpha T(n)}$ and $X_{T(n)}(jq_{T(n)})$, for arbitrary $\alpha>0$, can be verified by a straightforward use of the Normal Comparison Lemma. Condition (4.14) may be verified similarly. Finally, (4.15) follows from (5.30). Thus, by Proposition 4.3,

$$P\left\{\max_{0\leq jq_{T(n)}\leq T(n)} X_{T(n)}(jq_{T(n)})=0\right\}\to \exp(-\tau H_a(\alpha)). \tag{5.34}$$

As noted in Section 4, this is sufficient for (4.4), viz.

$$P\{X_{T(n)}([0,\ T(n)]^d)=0\}\to\exp(-\tau). \tag{5.35}$$

So, by Theorem 4.1 and the fact that $\{T(n)\}$ is arbitrary, the normalized set of exceedances of the level u_T by $\{Y_s\}$, i.e. the random set $\{s\in R_+^d,\ Y_{sT}\geq u_T\}$ converges in distribution to the support of some Poisson process on R_+^d with intensity τ.

It is not hard to prove that (5.33) holds for any family $\{u_T\}$ satisfying (5.26), provided (5.16) is replaced by either of

$$r(s)\log\|s\|\to 0 \quad \text{as } \|s\|\to\infty, \tag{5.36}$$

$$\int r(s)^2\,\mathrm{d}s<\infty. \tag{5.37}$$

Cf. [15, Ch. 12.5] for the case $d=1$.

Now suppose (5.36) or (5.37). Let F be a distribution function and write τ for the left-continuous version of $-\log F$. Put $c=\inf\{x,\ \tau(x)<\infty\}$. Proceed as in the discrete case to obtain increasing right-continuous functions c_T satisfying

$$c_T(y)\geq x \quad\text{iff}\quad y\geq u_T(\tau(x)), \qquad y\in R,\ x>c. \tag{5.38}$$

The sample path condition (4.3) follows as in the one-level case above. So does also (3.16) for arbitrary $\alpha>0$. From this case we also conclude (4.4). That (3.9) holds for all $\alpha > 0$ can be seen by rather straightforward calculations using the Normal Comparison Lemma and the fact that (5.33) holds for any family $\{u_T\}$ of levels satisfying (5.26). So, by Theorem 4.1, we conclude that the distribution of the semicontinuous process ξ_T, given by

$$\xi_T(s) = c_T(Y_{sT}), \quad s \in R_+^d, \tag{5.39}$$

converges, for $T \to \infty$, to the distribution of some semicontinuous process ξ on R_+^d with independent peaks satisfying

$$P\{\xi(K) \leqslant x\} = F(x)^{\lambda K}, \quad K \in \mathcal{K}. \tag{5.40}$$

Recall that this formula characterizes the distribution of ξ completely.

A case of particular interest arises when $\tau(x) = \mathrm{e}^{-x}$. Here we may choose $u_T(\tau(x)) = x/a_T + b_T$, where

$$a_T = (2 \log T^d)^{1/2}, \tag{5.41}$$

$$b_T = a_T + a_T^{-1}\left(\frac{2d-a}{2a} \log\log T^d + \log H_a - \tfrac{1}{2}\log 2\pi + \frac{2d-a}{2a}\log 2\right). \tag{5.42}$$

Cf. [3, 15].

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